

On the Classification of MDS Codes

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Abstract

A q -ary code of length n , size M , and minimum distance d is called an $(n, M, d)_q$ code. An $(n, q^k, n - k + 1)_q$ code is called a maximum distance separable (MDS) code. In this work, some MDS codes over small alphabets are classified. It is shown that every $(k + d - 1, q^k, d)_q$ code with $k \geq 3$, $d \geq 3$, $q \in \{5, 7\}$ is equivalent to a linear code with the same parameters. This implies that the $(6, 5^4, 3)_5$ code and the $(n, 7^{n-2}, 3)_7$ MDS codes for $n \in \{6, 7, 8\}$ are unique. The classification of one-error-correcting 8-ary MDS codes is also finished; there are 14, 8, 4, and 4 equivalence classes of $(n, 8^{n-2}, 3)_8$ codes for $n = 6, 7, 8, 9$, respectively. One of the equivalence classes of perfect $(9, 8^7, 3)_8$ codes corresponds to the Hamming code and the other three are nonlinear codes for which there exists no previously known construction.

I. INTRODUCTION

A CODE of length n over an alphabet \mathcal{A} is a subset of \mathcal{A}^n . With alphabet size $q = |\mathcal{A}|$, the code is called a q -ary code. The number of codewords is called the *size* of the code. The *Hamming distance* between two words in \mathcal{A}^n is the number of coordinates in which they differ. The *minimum distance* of a code is the minimum Hamming distance between any two distinct codewords. A code with minimum distance d is able to detect errors in up to $d - 1$ coordinates and correct errors in up to $\lfloor (d - 1)/2 \rfloor$ coordinates. A q -ary code of length n , size M , and minimum distance d is called an $(n, M, d)_q$ code.

A code with the alphabet \mathbb{F}_q , the finite field of order q , is *linear* if the codewords form a vector subspace of \mathbb{F}_q^n . For *unrestricted* (that is, either linear or nonlinear) codes, two codes are called *equivalent* if one can be obtained from the other by a permutation of coordinates followed by permutations of symbols at each coordinate separately. We use the notation $C \cong C'$ to denote that codes C and C' are equivalent. Equivalence maintains the Hamming distance between codewords but not linearity.

A general bound for the size of an $(n, M, d)_q$ code is the Singleton bound [1], which states that

$$M \leq q^{n-d+1}.$$

Codes with $M = q^{n-d+1}$ are called *maximum distance separable (MDS)*.

The Hamming bound, or the sphere-packing bound, states that

$$M \leq \frac{q^n}{\sum_{i=0}^t \binom{n}{i} (q-1)^i},$$

where $t = \lfloor \frac{d-1}{2} \rfloor$ is the number of errors a code with minimum distance d can correct. Codes attaining this bound are called *perfect*. For one-error-correcting codes, $d = 3$, and thus

$$M \leq \frac{q^n}{1 + n(q-1)}.$$

Even the existence of linear MDS codes with given parameters is in general an open question (see [2, Chapter 11]), and less is known about the unrestricted case. The $(n, q^2, n-1)_q$ codes correspond to sets of mutually orthogonal Latin squares, which have been widely studied [3]. For some results for other unrestricted MDS codes, see [4]–[7].

Perfect one-error-correcting MDS codes are $(q+1, q^{q-1}, 3)_q$ codes. For a prime power q , the only linear code up to equivalence with these parameters is the Hamming code, whose parity check matrix contains the maximal number $q+1$ of pairwise linearly independent columns. A natural question is whether codes with the same parameters exist that are not equivalent to linear codes.

The $(3, 2^1, 3)_2$ code is trivially unique, and the uniqueness of the $(4, 3^2, 3)_3$ code is not difficult to prove either. Alderson [8] showed that the $(5, 4^3, 3)_4$ code is unique. The nonexistence of Graeco-Latin squares of order 6 implies the nonexistence of $(7, 6^5, 3)_6$ codes. The cases $q = 5, 7, 8$ are settled in the present work: the $(6, 5^4, 3)_5$ and $(8, 7^6, 3)_7$ codes are unique and there exists four equivalence classes of $(9, 8^7, 3)_8$ codes.

In the general case, when q is a proper prime power and $q \geq 9$, there exists a $(q+1, q^{q-1}, 3)_q$ code that is not equivalent to the Hamming code with the same parameters, as demonstrated by an early construction by Lindström [9]. Heden [10] studied

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certain perfect codes when q is a prime and showed that they are equivalent to linear codes. There exist also constructions for nonlinear perfect codes using more restrictive notions of equivalence, such as [11].

Shortening the perfect codes gives one-error-correcting $(n, q^{n-2}, 3)_q$ MDS codes for $3 \leq n < q + 1$. Our work relies on known classification results of $(n, q^{n-2}, 3)_q$ MDS codes for $n = 4, 5$. For $n = 4$, the codes are equivalent to Graeco-Latin squares of order q , which have been classified for $q \leq 8$ by McKay [12]; there are 1, 1, 1, 0, 7, 2165 equivalence classes of such codes for $q = 3, 4, \dots, 8$, respectively. For $n = 5$, the codes are equivalent to Graeco-Latin cubes which have been classified recently [13]; there are 1, 1, and 12484 equivalence classes of such codes for $q = 5, 7, 8$, respectively.

This work consists of two parts. In the first part, we show that every $(k + d - 1, q^k, d)_q$ code, where $k, d \geq 3$ and $q = 5, 7$, is equivalent to a linear code. For one-error-correcting codes, this implies that the $(6, 5^4, 3)_5$ code and the $(n, 7^{n-2}, 3)_q$ codes for $n = 6, 7, 8$ are unique. This part is easier to carry out using the terminology of Latin squares. In the second part, we present an algorithm for exhaustive generation of $(n, q^{n-2}, 3)_q$ codes starting from $(n - 1, q^{n-3}, 3)_q$ codes. Running this algorithm for $q = 8$ yielded 14, 8, 4, and 4 equivalence classes of $(n, 8^{n-2}, 3)_8$ codes for $n = 6, 7, 8, 9$, respectively.

II. PRELIMINARIES

For ease of notation, we denote $[m] = \{1, 2, \dots, m\}$ when referring to sets of indices.

A. Latin Hypercubes and MDS Codes

A *Latin square* of order q is a $q \times q$ array of symbols from an alphabet \mathcal{A} of size q such that each symbol appears exactly once in each row and each column. Two Latin squares are called *orthogonal* if each pair of symbols occurs exactly once when the squares are superimposed. A pair of orthogonal Latin squares is called a *Graeco-Latin square*.

A *Latin hypercube* of dimension k is a $q \times q \times \dots \times q$ (k times) array of symbols from an alphabet \mathcal{A} of size q where each $q \times q$ subarray, obtained by fixing any $k - 2$ coordinates, is a Latin square. Two Latin hypercubes of same dimension are called orthogonal if when the hypercubes are superimposed, every $q \times q$ subarray is a Graeco-Latin square. A pair of Latin hypercubes is called a *Graeco-Latin hypercube*.

We denote the positions in a Latin hypercube of dimension k by elements in \mathcal{A}^k , so Latin hypercubes can be viewed as functions from \mathcal{A}^k to \mathcal{A} . For ease of notation, we assume that $\mathcal{A} = \mathbb{F}_q$ when q is a prime power unless otherwise mentioned.

There is a one-to-one correspondence between Latin hypercubes of order q and dimension k and $(k + 1, q^k, 2)_q$ codes: let $c = (c_1, c_2, \dots, c_{k+1})$ be a codeword if c_{k+1} occurs at position (c_1, c_2, \dots, c_k) in the Latin hypercube. Similarly, there is a one-to-one correspondence between Graeco-Latin hypercubes of order q and dimension k and $(k + 2, q^k, 3)_q$ MDS codes: let $c = (c_1, c_2, \dots, c_{k+2})$ be a codeword if (c_{k+1}, c_{k+2}) occurs at position (c_1, c_2, \dots, c_k) in the Graeco-Latin hypercube.

We define linearity of Latin hypercubes and tuples of Latin hypercubes as follows. A Latin hypercube f of order q and dimension k is linear if there are permutations $\alpha_0, \alpha_1, \dots, \alpha_k$ of \mathbb{F}_q such that

$$\alpha_0(f(x_1, x_2, \dots, x_k)) = \alpha_1(x_1) + \alpha_2(x_2) + \dots + \alpha_k(x_k). \quad (1)$$

This is equivalent to the condition that the corresponding MDS code be equivalent to a linear code. An r -tuple of (not necessarily mutually orthogonal) Latin hypercubes (f_1, f_2, \dots, f_r) is linear if there are permutations $\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_r$ of \mathbb{F}_q and coefficients $a_{i,j} \in \mathbb{F}_q$ for $i \in [r]$, $j \in [k]$ such that

$$\beta_i(f_i(x_1, x_2, \dots, x_k)) = a_{i,1}\alpha_1(x_1) + a_{i,2}\alpha_2(x_2) + \dots + a_{i,k}\alpha_k(x_k),$$

for each $i \in [r]$. We may assume that $a_{1,i} = 1$ for all i . For Graeco-Latin hypercubes, this is equivalent to the condition that the corresponding MDS code be equivalent to a linear code.

B. Properties of MDS Codes

Codes can be transformed into shorter and longer codes by operations called shortening and extending. Because these operations are used extensively in the description of the algorithm, we introduce precise notation for them here.

Definition II.1. For an $(n, M, d)_q$ MDS code C , let

$$s(C, i, v) = \{(c_1, c_2, \dots, c_{i-1}, c_{i+1}, \dots, c_n) : c \in C \text{ and } c_i = v\}.$$

This operation is called *shortening*.

Definition II.2. For an $(n, M, d)_q$ MDS code C , let

$$e(C, i, v) = \{(c_1, c_2, \dots, c_{i-1}, v, c_i, \dots, c_n) : c \in C\}.$$

This operation is called *extending*.

In other words, $s(C, i, v)$ is the $(n-1, M', d')_q$ code that is obtained by removing the i th coordinate from C and retaining the codewords that have v at that coordinate, and $e(C, i, v)$ is the $(n+1, M, d)_q$ code which is obtained by adding a coordinate at i with the symbol v to each codeword of C .

The following basic theorems are important in the construction of MDS codes based on shorter codes presented in Section IV.

Theorem II.3. *A shortened MDS code is an MDS code.*

Theorem II.4. *An $(n, q^k, n-k+1)_q$ MDS code is a union of q extended MDS codes: for each coordinate i there are $(n-1, q^{k-1}, n-k+1)_q$ MDS codes C'_v for each $v \in \mathbb{F}_q$ such that*

$$C = \bigcup_{v \in \mathbb{F}_q} e(C'_v, i, v).$$

Proof: Simply choose $C'_v = s(C, i, v)$. ■

C. Code Equivalence

The operations maintaining equivalence of codes of length n and alphabet \mathcal{A} form a group G that acts on \mathcal{A}^n . Each element $g \in G$ can be expressed in terms of a permutation π of $[n]$ and permutations σ_i for $i \in [n]$ of \mathcal{A} as

$$g = (\pi; \sigma_1, \sigma_2, \dots, \sigma_n),$$

such that for each $c = (c_1, c_2, \dots, c_n) \in \mathcal{A}^n$ and for each $i \in [n]$,

$$(gc)_{\pi(i)} = \sigma_{\pi(i)}(c_i),$$

where $(gc)_i$ denotes the i th symbol of gc .

Two codes, C and C' , are thus equivalent when there exists a $g \in G$ such that $C = gC'$. The set of all elements of G that map C to C' is denoted by $\text{Iso}(C, C')$. An element of $\text{Iso}(C, C)$ is called an automorphism of C . The group of automorphisms of C is denoted by $\text{Aut}(C)$. For equivalent codes C and C' , we can write

$$\text{Iso}(C, C') = \text{Aut}(C')g, \tag{2}$$

where g is any element of $\text{Iso}(C, C')$.

Each word that has value v at coordinate i is mapped by g to a word that has value $\sigma_{\pi(i)}(v)$ at coordinate $\pi(i)$. We also define an action of G on $[n] \times \mathcal{A}$ by

$$g(i, v) = (\pi(i), \sigma_{\pi(i)}(v)).$$

When the length of the codes is not obvious from the context, we denote by $G = G_n$ the group acting on \mathcal{A}^n . Because the study of equivalence of shortened codes of two codes plays a crucial role in the algorithm, we need the following two definitions to ease notation.

Definition II.5. *For every $g \in G_n$ and every $i \in [n]$, define $e(g, i) \in G_{n+1}$ to be the element that applies g to the subcodes obtained by removing i and keeps the coordinate i intact, that is,*

$$e(gC, i, v) = e(g, i)e(C, i, v),$$

for every $v \in \mathcal{A}$, and $C \subseteq \mathcal{A}^n$.

Definition II.6. *For every $g \in G_n$ and every $i \in [n]$ such that g maps coordinate i to itself and does not permute the symbols in coordinate i , define $s(g, i) \in G_{n-1}$ such that it applies g ignoring the coordinate i to codes of length $n-1$, that is,*

$$s(gC, i, v) = s(g, i)s(C, i, v),$$

for each $v \in \mathcal{A}$ and $C \subseteq \mathcal{A}^n$.

D. Computational Tools

To solve the problem of code equivalence computationally, we reduce it to the graph isomorphism problem. For each q -ary code C of length n , we define a labeled coloured graph as follows. The graph contains n copies of the complete graph with q vertices, colored with the first colour. For each codeword, the graph contains a vertex colored with the second color. From a vertex corresponding to codeword c , there is an edge to the v th vertex in the i th complete graph if and only if c has a value v at coordinate i .

Now two codes, C and C' , are equivalent if and only if their corresponding graphs, H and H' , respectively, are isomorphic. The permutation of coordinates corresponds to permutation of the complete graphs, and the permutations of symbols in each coordinate corresponds to permutation of vertices in each complete graph. Moreover, in a graph isomorphism mapping H to

H' , the permutation of the vertices of the first colour uniquely determines the permutation of the vertices of the second colour, so there is a direct correspondence between $\text{Iso}(C, C')$ and the set of graph isomorphisms from H to H' .

The software *nauty* [14] can be used to find canonical labelings of graphs, which then can be used to find a graph isomorphism between isomorphic graphs. In addition, *nauty* returns the automorphism of a graph. Along with (2), this allows finding the set $\text{Iso}(C, C')$ for two codes C and C' . We use *nauty* in the sparse mode with the random Schreier method enabled.

III. THEORETICAL RESULTS

In this section, we show that an r -tuple of Latin hypercubes of prime order and dimension k , where $r \geq 2$ and $k \geq 3$, is linear if each pair of Latin hypercubes of dimension 3 obtained by fixing $k - 3$ coordinates from two hypercubes of the tuple is linear. We start by showing that every Latin hypercube of prime order and dimension k , where $k \geq 4$, is linear if every Latin hypercube obtained from it by fixing one coordinate is linear.

Definition III.1. A rectangle of directions i and j ($i \neq j$) is a quadruple $(a = (a_1, a_2, \dots, a_k), b = (b_1, b_2, \dots, b_k), c = (c_1, c_2, \dots, c_k), d = (d_1, d_2, \dots, d_k))$ of elements of \mathbb{F}_q^k such that $a_i = b_i$, $c_i = d_i$, $b_j = c_j$, and $d_j = a_j$ and $a_l = b_l = c_l = d_l$ for all $l \in [k] \setminus \{i, j\}$.

Lemma III.2. For every linear Latin hypercube f of prime order q there is a unique function $\text{Rect}_f : \mathbb{F}_q^3 \rightarrow \mathbb{F}_q$ such that for every rectangle (a, b, c, d) ,

$$f(a) = \text{Rect}_f(f(b), f(c), f(d)).$$

Proof: Using the notation in (1), we find that

$$f(a) = \alpha_0^{-1}(\alpha_0(f(b)) - \alpha_0(f(c)) + \alpha_0(f(d))).$$

■

Lemma III.3. A linear Latin hypercube f of order q can be uniquely reconstructed from the function Rect_f and the values $f(x_1, x_2, \dots, x_k)$ where at most one of x_i is nonzero.

Proof: When x has $m \geq 2$ nonzero elements, the value $f(x)$ can be uniquely determined from the function Rect_f and the values $f(x')$ where x' has $m - 1$ nonzero elements using

$$\begin{aligned} f(x_1, \dots, x_i, \dots, x_j, \dots, x_k) \\ = \text{Rect}_f(f(x_1, \dots, 0, \dots, x_j, \dots, x_k), f(x_1, \dots, 0, \dots, 0, \dots, x_k), f(x_1, \dots, x_i, \dots, 0, \dots, x_k)). \end{aligned}$$

The lemma follows by induction on m .

■

Lemma III.4. Let f be a hypercube of dimension k , where $k \geq 4$, such that each $(k - 1)$ -dimensional Latin hypercube obtained from f by fixing one argument is linear. Then f is linear.

Proof: For $j \in [k]$, let r_j be the linear hypercube of dimension $k - 1$ obtained from f by letting the j th argument be 0. Without loss of generality, we may assume that

$$r_n(x_1, x_2, \dots, x_{k-1}) = x_1 + x_2 + \dots + x_{k-1},$$

and that

$$f(0, 0, \dots, 0, x_k) = x_k.$$

Now

$$\text{Rect}_{r_k}(a, b, c) = a - b + c.$$

For $j \in [k - 1]$, let s_j be the linear hypercube of dimension $k - 2$ obtained by letting the j th and the k th argument of f be 0. Because s_j occurs as a subarray in both r_j and r_k , we get

$$\text{Rect}_{r_j} = \text{Rect}_{s_j} = \text{Rect}_{r_k}.$$

Because

$$r_j(0, 0, \dots, 0, x_i, 0, \dots, 0) = x_i,$$

where $i \in [k - 1]$ and x_i occurs in the i th position, Lemma III.3 implies that

$$r_j(x_1, x_2, \dots, x_{k-1}) = x_1 + x_2 + \dots + x_{k-1},$$

for each $j \in [k]$, or equivalently,

$$f(x_1, x_2, \dots, x_k) = x_1 + x_2 + \dots + x_k, \tag{3}$$

when $x_i = 0$ for at least one value of i .

For each $a \in \mathbb{F}_q$, let t_a be the Latin hypercube obtained from f by letting the last argument be a . Now

$$t_a(0, 0, \dots, 0, x_i, 0, \dots, 0) = x_i + a,$$

where x_i occurs in the i th position. The function Rect_{t_a} is determined by (3), and again by Lemma III.3, we get that

$$t_a(x_1, x_2, \dots, x_{k-1}) = x_1 + x_2 + \dots + x_{k-1} + a,$$

for all a , or equivalently

$$f(x_1, x_2, \dots, x_k) = x_1 + x_2 + \dots + x_k.$$

Thus, f is linear. ■

We need one more lemma before proving the main theorem.

Lemma III.5. *Let q be a prime, let $c \in \mathbb{F}_q$, let $a_1, a_2, a_3 \in \mathbb{F}_q \setminus \{0\}$, and let γ_1, γ_2 , and γ_3 be permutations of \mathbb{F}_q . If*

$$\gamma_1(x_1) + \gamma_2(x_2) + \gamma_3(x_3) = c$$

whenever

$$a_1x_1 + a_2x_2 + a_3x_3 = 0,$$

then γ_i is an affine transformation of \mathbb{F}_q , for all i .

Proof: For all $x \in \mathbb{F}_q$, we find that

$$\gamma_1(x+1) - \gamma_1(x) = [c - \gamma_2(-a_2^{-1}a_1x) - \gamma_3(-a_3^{-1}a_1)] - [c - \gamma_2(-a_2^{-1}a_1x) - \gamma_3(0)] = \gamma_3(0) - \gamma_3(-a_3^{-1}a_1).$$

Because 1 generates the additive group of \mathbb{F}_q , we get

$$\gamma_1(x) = [\gamma_3(0) - \gamma_3(-a_3^{-1}a_1)]x + \gamma_1(0),$$

for each $x \in \mathbb{F}_q$. Thus, γ_1 is an affine transformation. By symmetry, so are γ_2 and γ_3 . ■

Theorem III.6. *Let (f_1, f_2, \dots, f_r) be an r -tuple of Latin hypercubes of prime order q and dimension k , with $r \geq 2$ and $k \geq 4$, such that each pair of Latin cubes obtained from any pair of them by fixing the same $k-3$ arguments is linear. Then (f_1, f_2, \dots, f_r) is a linear r -tuple of Latin hypercubes.*

Proof: By induction and Lemma III.4, f_i is a linear Latin hypercube for each i . Without loss of generality, we may assume that

$$f_i(x_1, x_2, \dots, x_k) = \gamma_{i,1}(x_1) + \gamma_{i,2}(x_2) + \dots + \gamma_{i,k}(x_k),$$

for each $i \in [r]$, where $\gamma_{i,j}$ are permutations of \mathbb{F}_q and $\gamma_{1,j}$ is the identity for each $j \in [k]$.

Consider some $i \in [r]$ and distinct $j_1, j_2, j_3 \in [k]$. Letting all arguments except j_1, j_2, j_3 of f_1 and f_i be 0, we obtain a linear pair (g, h) of Latin hypercubes of dimension 3 for which

$$\begin{aligned} \beta_0(g(x_1, x_2, x_3)) &= \beta_0(x_1 + x_2 + x_3) = \alpha_1(x_1) + \alpha_2(x_2) + \alpha_3(x_3), \\ \beta_1(h(x_1, x_2, x_3)) &= \beta_1(\gamma_{i,j_1}(x_1) + \gamma_{i,j_2}(x_2) + \gamma_{i,j_3}(x_3)) = a_1\alpha_1(x_1) + a_2\alpha_2(x_2) + a_3\alpha_3(x_3), \end{aligned}$$

for some $a_1, a_2, a_3 \in \mathbb{F}_q$ and permutations $\beta_0, \beta_1, \alpha_1, \alpha_2, \alpha_3$ of \mathbb{F}_q .

Because $\beta_0(g(x_1, x_2, x_3)) = \beta_0(0)$ whenever $x_1 + x_2 + x_3 = 0$, we see by Lemma III.5 that α_1, α_2 , and α_3 are affine transformations. Similarly, $h(x_1, x_2, x_3)$ is a function of $b_1x_1 + b_2x_2 + b_3x_3$ for some $b_1, b_2, b_3 \in \mathbb{F}_q$, and thus γ_{i,j_l} is an affine transformation for each $l \in \{1, 2, 3\}$.

Therefore, $\gamma_{i,j}$ is an affine transformation for all $i \in [r]$ and $j \in [k]$. Thus, (f_1, f_2, \dots, f_r) is a linear r -tuple of Latin hypercubes. ■

Using the known computational results for Graeco-Latin cubes of orders 5 and 7, Theorem III.6 implies the following.

Theorem III.7. *Every code with parameters $(k+d-1, 7^k, d)_7$ or $(k+d-1, 5^k, d)_5$, where $k, d \geq 3$, is equivalent to a linear code.*

Proof: For every $(n, q^k, d)_q$ code C with $n = k+d-1$, there is a $(d-1)$ -tuple of mutually orthogonal Latin hypercubes $(f_1, f_2, \dots, f_{d-1})$ of order q and dimension n such that C is the set of n -tuples (x_1, x_2, \dots, x_n) that satisfy

$$\begin{aligned} f_1(x_1, x_2, \dots, x_k) &= x_{k+1}, \\ f_2(x_1, x_2, \dots, x_k) &= x_{k+2}, \\ &\vdots \\ f_{d-1}(x_1, x_2, \dots, x_k) &= x_{k+d-1}. \end{aligned}$$

Because every Graeco-Latin cube of order 5 or 7 is linear, (f_1, \dots, f_k) is a linear $(d-1)$ -tuple of Latin hypercubes for $q = 5, 7$ by Theorem III.6. Therefore, C is equivalent to a linear code. ■

Corollary III.8 (MDS conjecture for $q = 5, 7$). *For $q \in \{5, 7\}$, $k \geq 2$ and $d = n - k + 1 > 2$, there exists an $(n, q^k, n - k + 1)_q$ MDS code if and only if $n \leq q + 1$.*

Proof: The case $k = 2$ follows from the well known theorem that the size of a set of mutually orthogonal Latin squares of order q is at most $q - 1$. We have shown that the existence of any MDS code for $k \geq 3$, $d \geq 3$ implies the existence of a linear code with the same parameters, and the MDS conjecture is true for linear codes over prime fields [15]. ■

Lemma III.9. *Let q be a prime power and $n \in \{q - 1, q, q + 1\}$. All linear $(n, q^{n-2}, 3)_q$ codes are equivalent.*

Proof: Let α be a primitive element of \mathbb{F}_q . After multiplying each column by a scalar, the parity check matrix of an $(n, q^{n-2}, 3)_q$ code can be written as

$$\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix},$$

where all a_i are distinct. Because at most two elements from \mathbb{F}_q are missing from $S = \{a_1, a_2, \dots, a_{n-1}\}$ when $n \geq q - 1$, there is an affine transformation $x \mapsto bx + c$ with $b \neq 0$ that maps S to $\{0, 1, \alpha^1, \alpha^2, \dots, \alpha^{n-2}\}$. Multiplying the second row by b , adding the first row multiplied by c to the first row, multiplying the first column by b^{-1} and permuting the columns yields

$$\begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \alpha^1 & \cdots & \alpha^{n-2} \end{pmatrix}.$$

Because elementary row operations on the parity check matrix do not change the code and multiplying a column and permuting columns maintain equivalence, every linear $(n, q^{n-2}, 3)_q$ code is equivalent to the code with the parity check matrix described above. ■

Corollary III.10. *The $(6, 5^4, 3)_5$ code and the $(n, 7^{n-2}, 3)_7$ codes for $n = 6, 7, 8$ are unique.*

Proof: By Theorem III.7 these codes are linear, and by Lemma III.9 they are equivalent. ■

IV. COMPUTATIONAL CLASSIFICATION

A. Algorithm

The algorithm to be presented generates representatives of all equivalence classes of $(n+1, q^{n-1}, 3)_q$ codes using an ordered set of representatives of equivalence classes of $(n, q^{n-2}, 3)_q$ codes, denoted by $\hat{S}^n = \{\hat{C}_1^n, \hat{C}_2^n, \dots, \hat{C}_N^n\}$. For simplicity, we assume that every \hat{C}_k^n contains the all-zero codeword.

Definition IV.1. *Let ϕ be a function that maps each $(n, q^{n-2}, 3)_q$ code C to an integer in $[N]$ such that $C \cong \hat{C}_{\phi(C)}^n$.*

To reduce the search tree and the number of equivalent codes generated, we construct only $(n+1, q^{n-1}, 3)_q$ codes and their subsets of a certain form. More precisely, we call a subset C of \mathbb{F}_q^{n+1} *semi-canonical* if it satisfies the following properties:

- 1) C has minimum distance 3,
- 2) $s(C, 1, 0) = \hat{C}_k^n$ for some k ,
- 3) For all $i \in [n+1]$ and $v \in \mathbb{F}_q$ for which $s(C, i, v)$ has q^{n-2} codewords, $\phi(s(C, i, v)) \geq k$.

Every $(n+1, q^{n-1}, 3)_q$ code C is equivalent to a code that satisfies these properties.

The central part of the algorithm is a procedure which, given an index k , a coordinate $i \in [n]$, and $v \in \mathbb{F}_q$, finds, up to a permutation of the values $\mathbb{F}_q \setminus \{0\}$ in the first coordinate, all possible $(n, q^{n-2}, 3)_q$ codes C for which

$$e(\hat{C}_k^n, 1, 0) \cup e(C, i+1, v)$$

is semi-canonical. A necessary condition is that

$$s(C, 1, 0) = s(\hat{C}_k^n, i, v). \quad (4)$$

The following theorem yields a way to exhaustively construct the codes C satisfying the above condition.

Definition IV.2. *For each $i \in [n]$ and $v \in \mathbb{F}_q$, let $h_{i,v} \in G_n$ be the element that applies the cyclic permutation $(1\ 2 \cdots i)$ to the coordinates and then swaps the values v and 0 in the first coordinate.*

Theorem IV.3. *Let \hat{C} be an $(n, q^{n-2}, 3)_q$ code and let \hat{D} be an $(n-1, q^{n-3}, 3)_q$ code. Let C be a code equivalent to \hat{C} for which $s(C, 1, 0) = \hat{D}$. Now C can be expressed as*

$$C = g'e(g, 1)h_{i,v}\hat{C},$$

where $g' \in G_n$ permutes the values $\mathbb{F}_q \setminus \{0\}$ in the first coordinate and keeps other coordinates intact, (i, v) is a coordinate-value pair, and $g \in \text{Iso}(s(h_{i,v}\hat{C}, 1, 0), \hat{D})$.

Proof: Let $g'' \in G_n$ such that $C = g''\hat{C}$. Let $(i, v) = g''^{-1}(1, 0)$. Because $g''h_{i,v}^{-1}(1, 0) = (1, 0)$, we can express $g''h_{i,v}^{-1}$ as

$$g''h_{i,v}^{-1} = g'e(g, 1, 0),$$

where g' permutes the nonzero values in the first coordinate and keeps other coordinates intact and $g = s(g''h_{i,v}^{-1}, 1) \in G_{n-1}$. We obtain

$$\hat{D} = s(C, 1, 0) = s(g''\hat{C}, 1, 0) = s(g'e(g, 1, 0)h_{i,v}\hat{C}, 1, 0) = gs(h_{i,v}\hat{C}, 1, 0),$$

and thus $g \in \text{Iso}(s(h_{i,v}\hat{C}, 1, 0), \hat{D})$. ■

The codes C satisfying (4) are now generated with the following algorithm. We loop over all $l = k, k+1, \dots, |\hat{S}^n|$ and all coordinate-value pairs (j, w) for which $s(\hat{C}_l^n, j, w) \cong s(\hat{C}_k^n, i, v)$. In each step, we loop over all $g \in \text{Iso}(s(h_{j,w}\hat{C}_l^n, 1, 0), s(\hat{C}_k^n, 1, 0))$ and consider the code

$$C = e(g, 1)h_{j,w}\hat{C}_l^n, \quad (5)$$

and report it if

$$e(\hat{C}_k^n, 1, 0) \cup e(C, i+1, v)$$

has minimum distance 3.

We generate the $(n+1, q^{n-1}, 3)_q$ codes in two phases. In the first phase, we consider codes containing the codewords that have a 0 in the first or the second coordinate. These codes are potential subsets of $(n+1, q^{n-1}, 3)_q$ codes. More precisely, we construct, for each k separately, the semi-canonical codes that are of the form

$$e(\hat{C}_k^n, 1, 0) \cup e(C, 2, 0),$$

where C has the property that for all $v \in \mathbb{F}_q$ there is a $w \in \mathbb{F}_q$ such that C contains the codeword $v00..0vw$. These codes form the seeds for the next phase. The permutation of the nonzero values in the first coordinate of C can be chosen to satisfy the last requirement, so the seeds can be constructed by the procedure described above. We perform isomorph rejection on the obtained seeds, since equivalent seeds would be augmented to equivalent codes.

In the second phase, we start from a seed

$$C = e(\hat{C}_k^n, 1, 0) \cup e(C', 2, 0)$$

and find all semi-canonical $(n+1, q^{n-1}, 3)_q$ codes that have C as a subset. These codes can be written in the form

$$\bigcup_{v \in \mathbb{F}_q} e(C''_v, 3, v),$$

where each C''_v is an $(n, q^{n-2}, 3)_q$ code with the following properties:

- $\phi(C''_v) \geq k$,
- $e(\hat{C}_k^n, 1, 0) \cup e(C''_v, 3, v)$ has minimum distance 3,
- $e(C', 2, 0) \cup e(C''_v, 3, v)$ has minimum distance 3.

The first two properties allow us to find all possible choices for the code C''_v using the procedure described above. The third property implies

$$s(C', 2, v) = s(C''_v, 2, 0),$$

which either rejects a code immediately or yields a unique permutation of the values in the first coordinate of C''_v . The requirement that $e(C', 2, 0) \cup e(C''_v, 3, v)$ have minimum distance 3 can also be used to reject some choices. When all possible choices for C''_v for each v have been generated, we loop over all sets of C''_v for $v \in \mathbb{F}_q$ and report

$$D = \bigcup_{v \in \mathbb{F}_q} e(C''_v, 3, v)$$

if it is semi-canonical.

Most time is spent using *nauty* to detect code equivalence, so an obvious way to optimize performance is to reduce the number of code equivalence instances that need to be solved. For example, detecting the equivalence class where each shortened code $s(\hat{C}_k^n, i, v)$ belongs needs to be done only when generating the codes of length n , and the results can be used when generating the codes of length $n+1$. In addition, when generating codes in (5), we can consider only one (j, w) from each orbit of the coordinate-value pairs in the automorphism group of \hat{C}_l^n .

TABLE I
AUTOMORPHISM GROUP ORDERS OF $(n, 8^{n-2}, 3)_8$ CODES

$n = 6$		$n = 7$	
$ \text{Aut}(C) $	#	$ \text{Aut}(C) $	#
1 536	3	16 384	1
2 048	1	24 576	1
3 072	1	65 536	2
4 096	5	86 016	1
12 288	3	98 304	1
516 096	1	196 608	1
		9 633 792	1
$n = 8$		$n = 9$	
$ \text{Aut}(C) $	#	$ \text{Aut}(C) $	#
393 216	1	25 165 824	1
688 128	1	44 040 192	1
786 432	1	50 331 648	1
308 281 344	1	22 196 256 768	1

TABLE II
DETAILS OF THE SEARCH

n	# of seeds	# of inequivalent seeds	# of codes	# of inequivalent codes	CPU time (hours)
6	122	107	21	14	15
7	15	9	9	8	49
8	9	6	6	4	340
9	4	4	4	4	1516

B. Results

The algorithm was run for the case $q = 8$ starting from the representatives of the 12484 equivalence classes of $(5, 8^3, 3)_8$ codes constructed in [13] and proceeding step by step to the $(9, 8^7, 3)_8$ codes. The search yielded 14, 8, 4, and 4 equivalence classes of $(n, 8^{n-2}, 3)_8$ codes for $n = 6, 7, 8, 9$, respectively. The orders of the automorphism groups of the codes are given in Table I. One of the equivalence classes of perfect codes correspond to the Hamming code, and the other three are new nonlinear codes for which no known construction exists; for example, the construction in [11] is equivalent to the linear code with the present definition of code equivalence. The nonlinear codes are presented in the Appendix.

We give in Table II, for each n separately, the number of seeds before and after isomorph rejection and the number of codes the inequivalent seeds were augmented to, again before and after isomorph rejection. The time required for the search for each n is also given and corresponds to one core of an Intel Xeon E5-2665 processor. The time for case n includes the search for seeds and augmenting seeds, isomorph rejection after both steps, and identifying the shortened codes of obtained $(n, q^{n-2}, 3)_q$ codes to detect whether the codes are semi-canonical. These results can also be used when generating $(n+1, q^{n-1}, 3)_q$ codes, so the time requirement of a step would be higher if no previous results were available.

C. Consistency Check

To check the consistency of the results given by the algorithm, we count for each k in two ways the number N_k of semi-canonical $(n+1, q^{n-1}, 3)_q$ codes C for which $s(C, 1, 0) = \hat{C}_k^n$.

The first count is obtained by detecting subcodes of the $(n+1, q^{n-1}, 3)_q$ codes obtained. For an $(n+1, q^{n-1}, 3)_q$ code C and an $(n, q^{n-2}, 3)_q$ code C' , let $S(C, C')$ be the number of pairs (i, v) such that $s(C, i, v) \cong C'$. Let \mathcal{S}_k be the set of obtained inequivalent $(n+1, q^{n-1}, 3)_q$ codes C for which $\min_{i,v} \phi(s(C, i, v)) = k$. Consider an arbitrary $C \in \mathcal{S}_k$. The size of the equivalence class of C is simply $|G_{n+1}|/|\text{Aut}(C)|$. The proportion of the codes C' in the equivalence class for which $s(C', 1, 0) \cong \hat{C}_k^n$ is $S(C, \hat{C}_k^n)/(q(n+1))$. Further, the proportion of those that have $s(C', 1, 0) = \hat{C}_k^n$ is $|\text{Aut}(C_k)|/|G_n|$. Therefore, the total number N_k becomes

$$N_k = \frac{|\text{Aut}(C_k)|}{|G_n|} \sum_{C \in \mathcal{S}_k} \frac{|G_{n+1}| S(C, \hat{C}_k^n)}{|\text{Aut}(C)| q(n+1)} = (q-1)! |\text{Aut}(C_k)| \sum_{C \in \mathcal{S}_k} \frac{S(C, \hat{C}_k^n)}{|\text{Aut}(C)|}.$$

On the other hand, the number N_k can be obtained by finding the number of different codes that would be generated by the algorithm if equivalent codes were not rejected at any phase of the algorithm. Let \mathcal{T}_k be the set of seeds obtained during the search starting from the code \hat{C}_k^n that were not rejected during the isomorph rejection. For each seed $D \in \mathcal{T}_k$, let $N(D)$ be the number of different seeds equivalent to D obtained during the search, and let $M(D)$ be the number of semi-canonical full codes that were obtained from the seed. Now the count becomes

$$N_k = (q-1)! \sum_{D \in \mathcal{T}_k} N(D) M(D).$$

Here, the factor $(q-1)!$ accounts for the permutations of $\mathbb{F}_q \setminus \{0\}$ in the first coordinate of the seed.

This check also alerts if the obtained full codes contain subsets equivalent to codes that should have been seeds but were not obtained during the search, or if any seeds that are equivalent to obtained seeds are missing.

APPENDIX PERFECT ONE-ERROR-CORRECTING 8-ARY MDS CODES

It turns out that every nonlinear $(9, 8^7, 3)_8$ code C has the property that there is a coordinate i such that $s(C, i, v)$ is equivalent to the linear $(8, 8^6, 3)_8$ code for each v . This allows us to present the nonlinear perfect codes in terms of these shortened codes.

Let α be a primitive element of \mathbb{F}_8 with $\alpha^3 + \alpha^2 + 1 = 0$. An element $x \in \mathbb{F}_8$ can be written as

$$x = a_2\alpha^2 + a_1\alpha + a_0,$$

where $a_0, a_1, a_2 \in \{0, 1\}$. We denote the element x by a number in $\{0, 1, \dots, 7\}$ whose binary representation is $a_2a_1a_0$.

Let $C' \subseteq \mathbb{F}_8^8$ be a linear code with the generator matrix

$$\begin{pmatrix} 1 & & & & & & & 1 & 1 \\ & 1 & & & & & & 1 & 2 \\ & & 1 & & & & & 1 & 3 \\ & & & 1 & & & & 1 & 4 \\ & & & & 1 & & & 1 & 5 \\ & & & & & 1 & & 1 & 6 \end{pmatrix}.$$

Now a nonlinear $(9, 8^7, 3)_8$ code C can be expressed as

$$C = \bigcup_{v \in \mathbb{F}_8} e(g^v C', i, v),$$

where $i \in \{1, 2, \dots, 9\}$ and $g^v = (\pi^v; \sigma_1^v, \sigma_2^v, \dots, \sigma_8^v) \in G_8$ for permutations π^v of $\{1, 2, \dots, 8\}$ and permutations σ_j^v of \mathbb{F}_8 as defined in Section II-C.

Selecting the coordinate i corresponds to permuting the coordinates of the perfect code, so we may choose for example $i = 1$. For each of the three nonlinear equivalence classes of $(9, 8^7, 3)_8$, one choice of these permutations to generate one representative is given in Table III. The permutations π^v of $\{1, 2, \dots, 8\}$ are expressed as $\pi^v(1)\pi^v(2) \cdots \pi^v(8)$ and the permutation σ_i^v of \mathbb{F}_8 as $\sigma_i^v(0)\sigma_i^v(1) \cdots \sigma_i^v(7)$.

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TABLE III
PERMUTATIONS FOR CONSTRUCTING THE NONLINEAR PERFECT $(9, 8^7, 3)_8$ CODES

v	0	1	2	3	4	5	6	7
π^v	12836457	12564378	12743685	12743685	12835764	12564378	12835764	12836457
σ_1^v	06452371	05371246	01436752	04653127	06715432	07426153	02476351	03624157
σ_2^v	03624157	02471536	07536142	05643721	06235417	07356124	07426153	01346725
σ_3^v	07163542	20635174	04652731	06235417	03142657	60541732	07613524	05731264
σ_4^v	01346725	05217346	03625741	06542317	06457123	01436752	03146275	70615324
σ_5^v	07623154	25734610	04326157	03652471	04617523	61327540	03564271	01736425
σ_6^v	02764351	01374625	40236157	30714265	02347615	07265413	05632174	03247165
σ_7^v	03624157	02476351	25347610	32175460	40723165	07531624	60235471	01346725
σ_8^v	06523147	01243675	02375641	03547216	67531240	04765312	13742560	42753610
π^v	12653478	12738654	12738645	12734586	12843576	12654387	12845367	12735468
σ_1^v	02314675	03746512	06247153	02641375	05634127	07253416	04375162	01543267
σ_2^v	04157362	06421753	07435216	01457632	04372615	03561247	05726314	03476521
σ_3^v	07526134	05367241	01672345	03216547	02751364	50413627	70165234	02135746
σ_4^v	01765423	04152637	02754631	07523461	01546732	01674523	01453276	40263751
σ_5^v	05172463	03567214	07462351	70236451	20617354	07321546	05176324	05361742
σ_6^v	02453716	06352174	10627534	01752634	05367124	25673140	05247136	02541673
σ_7^v	04253617	27345160	25176340	23567410	07264153	01327456	05736124	24731650
σ_8^v	01765423	40516273	05361427	06174253	34256170	02315764	75643120	04627315
π^v	12438765	12438765	12347658	12347658	12347658	12438765	12438765	12347658
σ_1^v	02164753	04635172	02471653	07563124	01735462	06573421	03712546	05326741
σ_2^v	06573421	03712546	04273651	07561342	03715264	07241365	01456237	05146723
σ_3^v	03712546	02164753	32671450	45736210	56423170	01456237	06573421	74165320
σ_4^v	04635172	41672350	02537461	03641725	07316542	64521730	35746120	06124357
σ_5^v	01273645	03754261	06527431	02341765	07213546	07465312	05621437	03164257
σ_6^v	05261473	01637245	30165724	10547263	70216435	06743152	03425716	40321576
σ_7^v	01456237	60754312	04631527	03257146	01372654	50236741	70423156	02745361
σ_8^v	07652431	02513647	02746351	04531627	06415732	05341276	01467325	05123476

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